Membrane fluids and Dirac string fluids

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May 4, 2005

Abstract

There are two different methods to describe membrane (string) fluids, which use different field content. The relation between the methods is clarified by construction of combined method.

Dirac membrane field appears naturally in new approach. It provides a possibility to consider new aspects of electrodynamics-type theories with electric and magnetic sources. The membrane fluid models automatically prohibit simulatenos existence of electric and magnetic currents.

Possible applications to the dark energy problem are mentioned.

1 Introduction

Recently various models of membrane (string) fluids are considered. These models describe continuous distributions of non-intersecting membranes (strings). Each space-time point belongs to world surface of one membrane, i.e. foliation of space-time is specified (to describe membrane fluid one has to specify foliation and density of membrane matter).

String fluid models are applicable in various realms of physics and cosmology. E.g. one can apply zero-brane fluid to describe dark energy (on dark energy see e.g. [1] and references wherein). Other applications are related to string theory and relativistic elasticity theory.

There are two different approaches to string fluids, which have different field contents. These approaches are not equivalent.

Both methods describes fluid of $(n-1)$-dimensional membranes (with n-dimensional world surfaces $\mathbf{V}_\varphi$) in D-dimensional space-time $\mathbf{M}$ by means of $(D-n)$-form J (membrane field intensity), which is Hodge dual of n-form current density $j = *^{-1}J$.

To specify foliation one need form J up to non-zero scalar factor.

The action, written in terms of J has the similar form for both methods

$$S = \int\limits_{\mathbf{M}} *L_0(\|J\|), \tag{1}$$

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where

$$\|J\| = \sqrt{\frac{1}{(D-n)!} J_{M_1 \ldots M_{D-n}} \, J^{M_1 \ldots M_{D-n}}}, \tag{2}$$

$\|J\|$ represents density of membrane media in space-time $\mathbf{M}$. Lagrangian L_0 is minus energy density in attendant frame. (We use $(-,+,+,\ldots,+)$ signature of space-time metric.)

Equation of state is described by relation

$$P_{\text{fluid}} = L_0 - L_0' \, \|J\|. \tag{3}$$

Here P_{fluid} is pressure in directions orthogonal to membrane world surface.

Energy-momentum tensor has the following form

$$T_{MN} = L_0 \, P_{MN} + P_{\text{fluid}} \, \bar{P}_{MN}. \tag{4}$$

P_{MN} is projector to world surface $\mathbf{V}_\varphi$ (surface $\varphi = \text{const}$), $\bar{P}_{MN} = g_{MN} - P_{MN}$ is projector to directions orthogonal to world surface.

Due to conservation of membrane current J is closed

$$\delta j = 0, \qquad \Leftrightarrow \qquad dJ = 0. \tag{5}$$

Form J has to satisfy the following condition (orthogonality to membrane world surface $\mathbf{V}_\varphi$)

$$J_{M_1 M_2 \ldots M_{D-n}} \, A^{M_1} = 0, \qquad \forall \; A \in T\mathbf{V}_\varphi. \tag{6}$$

Two methods differ by parametrization of J in terms of dynamical field.

The form J is closed, so the most obvious way is to consider membrane field in terms of $(n-1)$-form potential I (similar to non-linear electrodynamics)

$$J_I = dI. \tag{7}$$

In this approach the action has the following form

$$S_I[I] = \int\limits_{\mathbf{M}} *L_0(\|J_I\|), \tag{8}$$

This method was introduced independently by several authors, see e.g. [2, 3, 4] and references wherein. Conservation condition (5) is satisfied automatically. To satisfy condition (6) one has to restrict himself by certain class of solutions (it can be done by introduction of additional term to action, which impose appropriate constraint). These solutions are generalizations of potential flow in standard hydrodynamics.

Potential I could be considered as membrane field intensity for Dirac membrane (string) fluid (see section 6).

Another way was initially suggested (in the particular case of constant volume form $\Omega_{\mathbf{F}}$) to describe single membranes [5]–[12] (papers [11, 12] suggest dual interpretation, which correspond to use of $*J$ form instead of J). It was later geometrized and generalized to membrane fluids and other continuous media in [13]–[18]. Using this method one has to consider auxiliary $(D-n)$-dimensional space $\mathbf{F}$ (points $\phi \in \mathbf{F}$ numerate membranes $\mathbf{V}_\phi$) with volume form, which represents density of membrane media in auxiliary space $\mathbf{F}$,

$$\Omega_{\mathbf{F}} = f(\phi) \, d\phi^1 \wedge \ldots \wedge d\phi^{D-n}, \tag{9}$$

where ϕ^α, $\alpha = 1, \ldots, D-n$ are coordinates at $\mathbf{F}$, and mapping

$$\begin{aligned} &\varphi : \mathbf{M} \to \mathbf{F}, \\ &\varphi(X) = (\varphi^\alpha(X)). \end{aligned} \tag{10}$$

Membrane field intensity is defined by the following relation

$$J_\varphi = \varphi^* \Omega_{\mathbf{F}} = f(\varphi)\, d\varphi^1 \wedge \ldots \wedge d\varphi^{D-n}. \tag{11}$$

In this approach the action has the following form

$$S_\varphi[\varphi] = \int\limits_{\mathbf{M}} *L_0(\|J_\varphi\|). \tag{12}$$

Conservation condition (5) is satisfied automatically. Moreover, condition (6) is also satisfied automatically. Every solution of model (8), which satisfies condition (6), is also solution (potential flow) of model (12). Field equations of model (8) are field equations of model (12) with extra condition of flow potentiality. These potential flows admit also dual description with J replaced by $*K = *\frac{\partial L_0}{\partial J} = L_0' \frac{*J}{\|J\|}$.

The theory written in terms of fields φ has no linear dependence in equations of motion. So, the theory admits straightforward Hamiltonian formulation and formal quantization [14]. The other approach to quantize the theory is based upon the Nambu brackets (see [19] and references wherein).

Here we present a method, which involves both φ and I. It allows to reveal new symmetries in electrodynamics-type theories and membrane fluid models.

2 Free membrane fluids

To involve constraint

$$J_I = J_\varphi \qquad \Leftrightarrow \qquad dI = \varphi^* \Omega_{\mathbf{F}} \tag{13}$$

one can introduce the following action

$$S_{\varphi I}[\varphi, I, K] = \int\limits_{\mathbf{M}} \left\{ *L_0\left(\sqrt{(J_I, J_\varphi)}\right) + \frac{1}{2}(J_\varphi - J_I) \wedge *K \right\}. \tag{14}$$

Here

$$(J_I, J_\varphi) = \frac{1}{(D-n)!} (J_I)_{M_1 \ldots M_{D-n}}\, (J_\varphi)^{M_1 \ldots M_{D-n}}. \tag{15}$$

Field equations are

$$J_I = J_\varphi \tag{16}$$

(so, we use notation $J = J_I = J_\varphi$),

$$\delta\left(K - L_0' \frac{J}{\|J\|}\right) = 0, \tag{17}$$

$$(J_{(\alpha)}, \delta K) = 0, \tag{18}$$

here $\delta = *^{-1} d *$,

$$J_{(\alpha)} = (-1)^{\alpha+1}\, f(\varphi)\, d\varphi^1 \wedge \ldots \widehat{d\varphi^\alpha} \ldots \wedge d\varphi^{D-n}.$$

The term $\widehat{d\varphi^\alpha}$ has to be skipped.

If one exclude fields I and K from field equation, then the equations appear to be equivalent to field equations for action (12). So, actions (12) and (14) describe the same system in different ways.

According to [17] field K could be considered as analogue of momentum in mechanics (in this case J is analogue of velocity).

K and I admits gauge-like transformations

$$\begin{aligned} K &\to K+\delta\lambda, \\ I &\to I+d\mu. \end{aligned} \tag{19}$$

3 Interactions of membrane fluids

To introduce interaction to action (14) one could add to action extra terms. To preserve the symmetries of the theory the terms have to be written in terms of fields J_φ and I only.

For arbitrary action

$$S[\varphi] = \int\limits_{\mathbf{M}} L(J_\varphi), \tag{20}$$

which involves fields φ^α through J_φ only, the equaction of motion, produced by variation by φ^α, is of the following form

$$\left(J_{(\alpha)}, \delta\left[\frac{\partial L}{\partial J_\varphi}\right]\right) = 0. \tag{21}$$

If the function $L(J_\varphi)$ is written in the form $L(J_\varphi) = L_1(\|J_\varphi\|, J_\varphi)$, then field equation acquire the form

$$\left(J_{(\alpha)}, \delta\left[\frac{\partial L_1}{\partial\|J_\varphi\|}\frac{J_\varphi}{\|J_\varphi\|}\right] + \mathcal{F}\right) = 0, \tag{22}$$

$$\mathcal{F} = \delta\left[\frac{\partial L_1}{\partial J_\varphi}\right] \tag{23}$$

(If action is written in terms of J_φ, I and K, the equations of motion have the similar form.)

The field $\mathcal{F}$, the equations of motion involve, satisfies the condition

$$\delta\mathcal{F} = 0. \tag{24}$$

In section 4 field $\mathcal{F} = \pm * F$, so $\delta\mathcal{F} = 0 \quad\Leftrightarrow\quad dF = 0$. I.e. "electric" membrane fluid interacts with the field, which admits no "magnetic" currents.

In section 5 field $\mathcal{F} = \pm F$, so $\delta\mathcal{F} = 0 \quad\Leftrightarrow\quad \delta F = 0$. I.e. "magnetic" membrane fluid interacts with the field, which admits no "electric" currents.

4 Membrane fluid as electric-type source

One can easily introduce interaction of membrane fluid with closed form F to action (14) by inserting standard interaction term

$$S_{\mathrm{el.}}[\varphi, I, K, A] = \int\limits_{\mathbf{M}} \left\{ *L_0(\sqrt{(J_I, J_\varphi)}) + \frac{1}{2}(J_\varphi - J_I) \wedge *K - \frac{1}{2} F \wedge *F + (-1)^{n+1} A \wedge J_I \right\}, \tag{25}$$

where $F = dA$ is closed $(n+1)$-form. So, $dF = 0$. Interaction term could be written in other equivalent form

$$\begin{aligned}(-1)^{n+1} \int\limits_{\mathbf{M}} A \wedge J_I = (-1)^{n+1} \int\limits_{\mathbf{M}} A \wedge dI = \int\limits_{\mathbf{M}} [-d(A \wedge I) + dA \wedge I] = \\ = - \int\limits_{\partial \mathbf{M}} A \wedge I + \int\limits_{\mathbf{M}} dA \wedge I.\end{aligned}$$

Up to surface term

$$S_{\mathrm{el.}}[\varphi, I, K, A] = \int\limits_{\mathbf{M}} \left\{ *L_0(\sqrt{(J_I, J_\varphi)}) + \frac{1}{2}(J_\varphi - J_I) \wedge *K - \frac{1}{2} F \wedge *F + F \wedge I \right\}, \tag{26}$$

Field equations are

$$J_I = J_\varphi \tag{27}$$

(we use notation $J = J_I = J_\varphi$),

$$\delta F = *^{-1} J, \tag{28}$$

$$\delta \left(K - L_0' \frac{J}{\|J\|} \right) + 2 \,\mathrm{sgn}(g) (-1)^{D-n-1} * F = 0, \tag{29}$$

$$\left(J_{(\alpha)}, \delta \left(L_0' \frac{J}{\|J\|} \right) + \mathrm{sgn}(g) (-1)^{D-n} * F \right) = 0. \tag{30}$$

K and I still admits gauge-like transformations (19).

One could derive from field equation (29), which is produced by variation of action by I, the condition

$$dF = 0.$$

Consequently the action (25), (26) does not allow any method of introduction of magnetic charges, which does not involve field I.

5 Membrane fluid as magnetic-type source

To introduce interaction of magnetic type with field of form F one can use method similar to Dirac strings. This method is partially similar to dual variable electrodynamics (see [20, 21] and references wherein).

The appropriate action has the form

$$S_{\mathrm{mag.}}[\varphi, I, K, A] = \int\limits_{\mathbf{M}} \left\{ *L_0(\sqrt{(J_I, J_\varphi)}) + \frac{1}{2}(J_\varphi - J_I) \wedge *K - \frac{1}{2} F \wedge *F \right\}, \tag{31}$$

where $(D-n-1)$-form F is defined by

$$F = dA + I. \tag{32}$$

Here field I represents Dirac string (membrane) distribution.

$$dF = J_I. \tag{33}$$

Field equations are

$$J_I = J_\varphi \tag{34}$$

(we use notation $J = J_I = J_\varphi$),

$$\delta F = 0, \tag{35}$$

$$\delta\left(K - L_0' \frac{J}{\|J\|}\right) + 2(-1)^{D-n} F = 0, \tag{36}$$

$$\left(J_{(\alpha)}, \delta\left(L_0' \frac{J}{\|J\|}\right) + (-1)^{D-n-1} F\right) = 0. \tag{37}$$

K still admits gauge-like transformations (19). Gauge-like transformation of I now involves potential A

$$\begin{aligned} I &\to I + d\mu, \\ A &\to A - \mu. \end{aligned} \tag{38}$$

This transformation (*generalised gradient transformation*) generalizes standard gauge transformation for field A. It corresponds to deformation of Dirac strings (membranes) with fixed boundaries.

By transformation (38) one can set $A = 0$ (i.e. $F = I$). Moreover, one can set $A = 0$ *before* variation of action.

$$S_{\mathrm{mag.}}[\varphi, I, K] = \int\limits_{\mathbf{M}} \left\{ *L_0(\sqrt{(J_I, J_\varphi)}) + \frac{1}{2}(J_\varphi - J_I) \wedge *K - \frac{1}{2} I \wedge *I \right\}, \tag{39}$$

It is possible because field equation (35), which is produced by variation of action by A, could be derived from field equation (36), which is produced by variation of action by I.

Consequently the action (31), (39) does not allow any method of introduction of electric charges, which does not involve field I.

6 Dirac strings (membranes)

This section just formulate standard Dirac string approach to introducing of magnetic charges to electrodynamics-type theories in the context of the paper. Theories considered below have quadratic Lagrangians, but the approach is also applicable to non-linear theories e.g. to Born-Infeld model.

We wrote the word “string” with quotation marks to emphasize that it is actually q-dimensional membrane.

The standard electrodynamics-type action has the following form

$$S[A(x)] = -\int\limits_{\mathbf{M}} \left\{ \frac{1}{2} F \wedge *F + A \wedge *j_e \right\}, \tag{40}$$

where

$$F = dA \tag{41}$$

is $(q+1)$-form field intensity, $\mathbf{M}$ is D-dimensional space-time region, A is q-form potential, j_e is q-form "electric" current density.

External derivative of equation (41) produce "first pair" of Maxwell-type equations

$$dF = 0. \tag{42}$$

Variation of action (40) by A produce "second pair" of Maxwell-type equations

$$\delta F = (-1)^q j_e. \tag{43}$$

Current j_e is conserved

$$\delta j_e = 0 \qquad \Leftrightarrow \qquad d * j_e = 0, \tag{44}$$

so action (40) is invariant under gradient transformations

$$A \longrightarrow A + df, \tag{45}$$

where f is arbitrary $(q-1)$-form.

To introduce "magnetic" currents one introduces Dirac "strings" $\mathbf{I}$ (open membranes with $(q+1)$-dimensional world surfaces $\mathbf{I}$, q-dimensional boundaries of membranes $\partial\mathbf{I}$ are world surfaces of magnetic sources) and consider action (40) with integration in region $\mathbf{M}\backslash\mathbf{I}$ with cut-off $\mathbf{I}$ instead of original region $\mathbf{M}$. At world surface $\mathbf{I}$ one has continuity condition for field intensity F

$$\exists \lim_{x \to x_0 \in \mathbf{I}\backslash\partial\mathbf{I}} F(x). \tag{46}$$

Dirac strings are similar to infinitely thin tube, which transfer magnetic flux from monopole to infinity or to other monopole of opposite sign. It allows to replace cut-off $\mathbf{I}$ and condition (46) by redefinition of field F

$$F = dA + I, \tag{47}$$

where I compensates magnetic flux inside tube.

New "first pair" of Maxwell-type equations is

$$dF = dI \qquad \Leftrightarrow \qquad \delta *^{-1} F = *^{-1} dI = j_m. \tag{48}$$

Here magnetic current

$$j_m = *^{-1} dI \tag{49}$$

is defined. By definition (compare with (53))

$$\delta j_m = *^{-1} d^2 I = 0. \tag{50}$$

One can use the old form of action (40) with new definition of F (47) and reproduce the same "second pair" of Maxwell-type equations (43) by variation of potential A.

If world surface $\mathbf{I}$ is defined by the following system

$$\begin{array}{rcll} b^\alpha(x)|_{\mathbf{I}} & = & 0, & \alpha = 1, \dots, q+1, \\ b^0|_{\mathbf{I}}(x) & \geq & 0, & \end{array} \tag{51}$$

and describes "magnetic" charge Q_m (numeration of functions b^α has to be in agreement with surface orientation), then

$$I = Q_m \theta(b^0(x))\, d\theta(b^1(x)) \wedge \ldots \wedge d\theta(b^{q+1}(x)), \tag{52}$$

where θ is Heaviside theta-function.

Dirac "string" describes singular (like point particles) magnetic currents with world lines (surfaces) $\partial\mathbf{I}$. Magnetic currents are conserved due to the following property of ∂ operation

$$\partial^2 \mathbf{I} = 0 \qquad \Leftrightarrow \qquad d^2 I = 0. \tag{53}$$

Dirac "string" has to go through points with zero j_e (to avoid interaction of magnetic flux inside tube with electric currents). To guarantee the possibility to find such surface one could require electric currents to be singular (like point particles) too.

Dirac "string" world surface is uniquely defined by $(q+1)$-form I, so actually we do not need Dirac "string" itself, but form I (Dirac "string" density) only. This point of view is more flexible, one has no longer obligation to restrict himself by fields I of form (52). It reveal more general symmetry like (38).

Form $J_I = dI$ is membrane field intensity. It satisfies conditions (5) and (6).

Form I is not unique. It is defined up to gradient transformation (38). By appropriate choice of gauge form I could satisfy condition (6) (with $\mathbf{V}_\varphi$ replaced by world surface $\mathbf{I}$ of Dirac membrane), but not (5). So form I is similar to non-conserved membrane field intensity. It describes *Dirac membrane fluid*. It is natural to replace condition (6), for field I, which holds only in special gauge by more subtle condition

$$dI_{M_1 M_2 \dots M_{D-n}}\, A^{M_1} = 0, \qquad \forall\ A \in T\mathbf{V}_\varphi. \tag{54}$$

7 Conclusion

7.1 Models variety

The approach proposed here is applicable not only to introduce interaction between membrane fluid and electrodynamics-type field. Instead of membrane fluid one can use membrane elastic media (see [18]), which also involve membrane density form J_φ. E.g. one could construct different classical models of elastic electrons.

The form of action of electrodynamics-type field is not fixed too. The method is also applicable for nonlinear models like Born-Infeld-type models, etc.

Even membrane fluid models are able to reproduce wide range of state equations. State equation $p(\rho)$ is related with Lagrangian $L_0(\|J\|) = -\rho(\|J\|)$ by the following relation:

$$\ln \|J\| = \int \frac{d\rho}{\rho + p(\rho)}. \tag{55}$$

E.g. state equation

$$p = w\rho \tag{56}$$

is reproduced by action

$$S_w[\varphi] = -\int\limits_{\mathbf{M}} *\|J_\varphi\|^{1+w}, \quad \text{or} \quad S_w[\varphi, I, K] = \int\limits_{\mathbf{M}} \left\{ *(J_I, J_\varphi)^{\frac{1+w}{2}} + \frac{1}{2}(J_\varphi - J_I) \wedge *K \right\}. \tag{57}$$

$\rho = -L_0 = \|J_\varphi\|^{1+w}$, see equations (3), (4).

Zero-brane fluid with negative pressure $-1.5 < w < -0.5$ could be useful to describe dark energy [1].

7.2 No-classical monopole hypothesis

The approach described has an unexpected heuristic issue. One can conclude that simultaneous existence of electric and magnetic currents is not possible. The electric-type sources considered in section 4 automatically suggests non-existence of magnetic type source (as consequence of field equations). Similarly, the magnetic-type sources considered in section 5 automatically suggests non-existence of electric type source (as consequence of field equations).

This difficulties have simple physical interpretation. The Dirac string approach require the possibility to draw Dirac string through the regions with no electric charges. If charges are point-like, then one able to draw Dirac strings in any situation. This requirement is crucial, because Dirac strings are similar to magnetic flux tubes. If a point electric charge intersect Dirac string it has to suffer infinitely large impact of Lorentzian force. Fortunately in classical theory with point particles the probability of this process is zero.

Nevertheless the classical theory with point particle is not self-consistent. If one replace point electric charges by continuous distributions, then there are configurations of charges, which make impossible to draw Dirac strings. Similarly, if magnetic charges are delocalized, then instead of single Dirac strings one has Dirac string distribution, which occupies finite volume.

In quantum theory one could expect delocalization of classical singular charges due to uncertainty principle. So, infinite impact with infinitesimal probability could produce the finite effect of interaction of electrical charges with Dirac strings.

The arguments above suggests the hypothesis, that there is no self-consistent classical theory with action, which involves both electric charges and magnetic monopoles.

Acknowledgement

The author is grateful to I.V. Volovich and I.Ya. Aref'eva. The work was partially supported by grants RFFI 02-01-01084, NSh-1542.2003.1, and by joint grant Y2-PM-11-08 of CRDF and RF Ministry of Education and Science.

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